UNIVERSITY OF BERN

Macroeconomics and International Economics

Putting Home Economics in Macroeconomics

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Contents

List of Tables

1 Introduction

Almost all macroeconomic models until the 1990s focused mainly on production in the market and neglected production in a home. This changed with the papers by Greenwood and Hercowitz (1991) and Greenwood et al. (1993) whose authors argue that neglecting production in the home is a missed opportunity in then modern macroeconomic models. A special emphasis lays on the circumstance that it's relatively easy to include house production in a macro-model. Furthermore, data from the 1990s underline their argument that house production must be an essential part of an economic model.

Greenwood et al. (1993) refer to data from the *Michigan Time use* survey which measured how a household allocates its time between working in the market, working at home and leisure. Whilst a household spends 33 percent working in the market, working at home follows with 25 percent of their remaining day time. Another economic aspect that shows the effect of house production on the gross national product can be found in the amount spent on business capital versus the amount spent on household capital. Where the second exceeds the first by nearly 15 percent (Greenwood et al. 1993). Overall the estimates for the household's sector output of the GDP range between 20 and 50 percent, depending on the survey.

Overall, the figures above illustrate that leaving out house production in a macroeconomic model seems like a missed opportunity. The model by Greenwood et al. (1993) includes a household production function into a standard real business cycle model (RBC). Following that households in this enriched model now face decisions on how to allocate their time not only between market work and leisure but between market work, leisure, and work at home. At the same time, a household must decide how to use the output of an economy by either consuming, investing in business capital, or investing in home capital. One example given by Greenwood and Hercowitz (1991) shows how home production may look like. For this, they take the example of preparing a meal at home. This process includes food produced in the market by hours spent working in the market as well as business capital, combined with home cooking services that use capital and time at home. A household then creates the end good: utility.

In the following sections 2-9, we derive the model by Greenwood et al. (1993) by presenting the integral parts of the models such as agents, the law of motion of capital and technology followed by a summary of the variables and parameters and finishing this part with an overview of the first-order conditions. Afterwards, we show in section 10 the deflating of the model, in section 11, we derive the deterministic steady state and in section 12 the log-transformation. In section 13, we comment on the use of the impulse response functions generated by Dynare. Section 14 provides an overview of the relevant equations of the model. Finally, in section 15, we add two extensions in form of minimal home production and a more general home production function.

2 Representative Household

In this model, there are three key agents, one of them being households. Greenwood et al. (1993) assume one representative household which maximizes its utility. The household faces decisions about how to allocate its time as well as how to allocate the economy's output along with consumption, investment in business capital, and investment in household capital. The household maximizes the following utility function:

$$U = \sum_{t=0}^{\infty} \beta^t [b \log(C_t) + (1-b) \log(l_t)] \tag{1}$$

A household gains utility from consumption (C_t) at period t as well as leisure time in said period (l_t). Where leisure time is given by the difference of all available time (equal to 1) minus hours worked in the market, h_{Mt}, and hours worked in the home, h_{Ht}:

$$l_t = 1 - h_{Mt} - h_{Ht} \tag{2}$$

A household's consumption comes from two sources, firstly from consuming goods that are produced in the market, c_{Mt}, and secondly from consuming goods and services produced in a home c_{Ht}, where e measures the willingness of a household to substitute between c_{Mt} and c_{Ht}, the larger e the greater is the willingness to substitute:

$$C_t = [a c_{Mt}^e + (1-a) c_{Ht}^e]^{\frac{1}{e}} \tag{3}$$

The two last constraints concern the allocation of output produced in the market and the home production function. Equation (4) is referred to as the *market budget constraint*. As the names suggest the constraint shows the decision faced by the household on how to allocate over its uses. The after-tax income given by the right-hand side can be used for consumption of market goods (c_{Mt}), investment in business capital (x_{Mt}) and/or investment in household capital (x_{Ht}).

$$c_{Mt} + x_{Mt} + x_{Ht} = w_t(1-\tau_h) h_{Mt} + r_t(1-\tau_k) k_{Mt} + \delta_M \tau_k k_{Mt} + T_t \tag{4}$$

Equation (5), which is the *home production function*, shows the constraint that a household faces whilst being active in home production. The household production function combines labor hours spent working at home (h_{Ht}) together with a technology variable (z_{Ht}) and household capital (k_{Ht}) to produce household output c_{Ht}.

$$c_{Ht} = g(h_{Ht}, k_{Ht}, z_{Ht}) = k_{Ht}^{\eta}(z_{Ht}h_{Ht})^{1-\eta} \tag{5}$$

Bear in mind that the output of house production can only be consumed in a household and not be invested in any kind of capital, referring back to the case of preparing a meal at home shows for example that a meal could not be invested back, neither in business capital nor in household capital.

3 Representative Firm

There is one representative firm in the model that features a constant return-to-scale Cobb-Douglas technology. The output of the firm is given by its production function which depends on hours worked in the market (h_{Mt}), the business capital stock (k_{Mt}) as well as on the available technology (z_{Mt}).

$$y_t = k_{Mt}^{\theta}(z_{Mt}h_{Mt})^{1-\theta} \tag{6}$$

The firm then maximizes profits by choosing the input factors k_{Mt} and h_{Mt}.

$$\max_{k_{Mt},h_{Mt}} \Pi_t = y_t - r_t k_{Mt} - w_t h_{Mt} \tag{7}$$

$$s.t. \qquad y_t = k_{Mt}^{\theta}(z_{Mt}h_{Mt})^{1-\theta} \tag{8}$$

4 Government

The model by Greenwood et al. (1993) also features a government component. The government receives income from taxing market wages ($w_t\tau_h h_{Mt}$) and return on market capital ($r_t\tau_k k_{Mt}$) minus a depreciation allowance ($\delta_M\tau_k k_{Mt}$). For simplicity, Greenwood et al. (1993) assume that the income of the government is redistributed entirely via a lump-sum transfer T_t back to the households, resulting in $G_t = 0$. Combining all the derived assumptions, we arrive at the government spending equation:

$$G_t = w_t\tau_h h_{Mt} + r_t\tau_k k_{Mt} - \delta_M\tau_k k_{Mt} - T_t = 0 \tag{9}$$

5 Capital

Investment augments the capital stock according to the following law of motion:

$$k_{t+1} = (1-\delta_M)k_{Mt} + (1-\delta_H)k_{Ht} + x_t \tag{10}$$

where $x_t = x_{Mt} + x_{Ht}$ is total investment. The aggregate capital stock can be divided between business (or market) and household capital at a point in time according to $k_t = k_{Mt} + k_{Ht}$. We assume that capital can be freely transformed between the home and market, although it may depreciate at different rates in the two sectors. Therefore, investments in the two capital goods are defined by

$$x_{Mt} = k_{Mt+1} - (1-\delta_M)k_{Mt} \tag{11}$$

$$x_{Ht} = k_{Ht+1} - (1-\delta_H)k_{Ht} \tag{12}$$

6 Technology

The evolution of technology in the model by Greenwood et al. (1993) consists of two parts split up between the "market" world and the "home production" world. First, z_{Mt} and z_{Ht} are shocks representing technological changes in the market and the home respectively. Both variables grow at the same rate, such that we end up with $z_{Mt} = \lambda^t \tilde{z}_{Mt}$ and $z_{Ht} = \lambda^t \tilde{z}_{Ht}$. The second part of technology evolution is given by so-called innovations ϵ_{Mt+1} and ϵ_{Ht+1}; the first concerns market technology whilst the second affects home technology. Summing up, we arrive at the following evolution of technology, where ρ_M and ρ_H are parameters that measure the persistence of the respective shocks.

$$\log(\tilde{z}_{Mt+1}) = \rho_M \log(\tilde{z}_{Mt}) + \epsilon_{Mt+1}, \quad \epsilon_{Mt} \sim N(0, \sigma_M^2) \tag{13}$$

$$\log(\tilde{z}_{Ht+1}) = \rho_H \log(\tilde{z}_{Ht}) + \epsilon_{Ht+1}, \quad \epsilon_{Ht} \sim N(0, \sigma_H^2) \tag{14}$$

Note: In equations (13) and (14), we used the deflated values of the technology shocks, denoted by tildes above the variables.

The variance of the technology shock is defined such that $\log(\tilde{z}_{Mt}^{1-\theta})$ and $\log(\tilde{z}_{Ht}^{1-\theta})$ have a standard deviation of 0.007. By taking the log of $\tilde{z}_{Ht}$ and plugging in the LHS of (14), we obtain the following expression:

$$\log(\tilde{z}_{Ht}^{1-\theta}) = (1-\theta)\log(\tilde{z}_{Ht}) = (1-\theta)\rho\log(\tilde{z}_{Ht-1}) + (1-\theta)\epsilon_{Ht} \tag{15}$$

This implies $(1-\theta)\epsilon_{Ht} \sim N(0, \overbrace{(1-\theta)^2\sigma_H^2}^{=0.007^2})$, we can then derive $\sigma_H = \frac{0.007}{1-\theta}$. The same arguments remain valid for the standard deviation σ_M.

7 Variables and Parameters

In this section, we present endogenous variables for the non-deflated general equilibrium in Table 1, the exogenous variables in Table 2, and the parameters in Table 3. We will update the reader during the text if the variables are denoted in levels or logs or if the variables are deflated or not.

Table 1: Endogenous Variables

	Meaning
[a]C	Total consumption
[a]c_H	Goods and services produced in the home
[a]c_M	Goods and services purchased in the market
[b]h_H	Labour hours spent working in the household
[b]h_M	Labour hours spent working in the market
[b]l	Leisure time $(1 - h_H - h_M)$
[c]k	Total capital
[c]k_H	Household capital
[c]k_M	Market capital
[a]r	Price at which business capital can be rented to firms
[b]T	Lump-sum transfer payment from the government
[b]w	Real wage rate in the market
[b]x	Total investment
[b]x_H	Investment in household capital
[b]x_M	Investment in business capital
[b]y	Market output
[c]z_H	Technology level in the home
[c]z_M	Technology level in the market
[c]$\tilde{z}_H$	Shock resulting from technological changes in the home
[c]$\tilde{z}_M$	Shock resulting from technological changes in the market

Note: In the deflated model the technology level equals the shock resulting from technology changes.

[a] denotes forward-looking variables (jumpers)

[b] denotes static variables

[c] denotes state variables

The innovations ϵ_{Mt} and ϵ_{Ht} are independent and identically distributed over time (see section 6) and feature a simultaneous correlation of γ.

Table 2: Exogenous Variables

	Meaning	Standard deviation
ϵ_H	Innovations in the home	σ_H
ϵ_M	Innovations in the market	σ_M

Note: See section 6 for the derivation of σ_M and σ_H.

The last table contains all parameters of this model. Note, that the parameter $e < 1$ controls the household's willingness to substitute between c_{Mt} and c_{Ht} the larger is e, the greater this willingness to substitute market consumption for home consumption. Similar for $\Psi < 1$, which describes the willingness to substitute between capital and labor in the general home production function stated in section 15.2.

Table 3: Parameters

	Meaning	Parameter value
a	Share of c_{Mt} of total consumption	*see footnote*
b	Weight factor of consumption vis-a-vis leisure	*see footnote*
e	Willingness of a household to substitute between market consumption c_{Mt} and home consumption c_{Ht}	$^{a}0$, $^{b/d}\frac{2}{3}$,$^{c}0.4$
β	Discount factor	0.9898
δ_H	Depreciation rate on household capital	0.0235
δ_M	Depreciation rate on business capital (tax-deductible)	0.0235
η	Capital share in the home production function	0.3245
γ	Measures the household's incentive, to move economic activity between the home and the market	$^{a/b}\frac{2}{3}$, $^{c}0$, $^{d}0.99$
ρ_H	Persistence of market technology shock	0.95
ρ_M	Persistence of home technology shock	0.95
σ_H	Standard deviation of innovations in the household	$\frac{0.007}{1-\theta}$
σ_M	Standard deviation of innovations in the market	$\frac{0.007}{1-\theta}$
τ_k	Tax rate on capital income	0.70
τ_h	Tax rate on labour income	0.25
θ	Capital share in the market production function	0.2944
λ	Growth rate of **all** endogenous variables besides h_{Mt}, h_{Ht}, l_t and r_t	1.004674
Ψ	Willingness of a household to substitute between capital k_{Ht} and time h_{Ht} in the home production	^{d}-0.5017

Note: We will determine a and b in dependence of the steady state values. λ is determined such that it matches the quarterly growth rate of output in the U.S. Data.

[a] Model with home production minimized

[b] Model with increased willingness to substitute between home and market

[c] Model with increased incentive to substitute between home and market

[d] Model with a more general home production function and highly correlated technology shocks

8 Maximization Problems

8.1 Household

The household chooses h_{Mt}, h_{Ht}, k_{Ht}, and k_{Mt} to maximize lifetime utility, subject to the constraints given below.

$$
\begin{aligned}
\max_{h_{Ht},h_{Mt},k_{Ht},k_{Mt}} \quad U &= \sum_{t=0}^{\infty} \beta^t \underbrace{(\frac{b}{e}\log(C_t^e) + (1-b)\log(l_t))}_{u(C_t,l_t)=u(c_{Mt},c_{Ht},h_{Mt}.h_{Ht})} \\
s.t. \qquad C &= (ac_{Mt}^e + (1-a)c_{Ht}^e)^{\frac{1}{e}} \\
l_t &= 1 - h_{Ht} - h_{Mt} \\
c_{Mt} &= w_t(1-\tau_h)h_{Mt} + r_t(1-\tau_k)k_{Mt} + \delta_M \tau_k k_{Mt} \\
&\quad + T_t - x_{Mt} - x_{Ht} \\
c_{Ht} &= \underbrace{k_{Ht}^{\eta}(z_{Ht}h_{Ht})^{1-\eta}}_{=g(h_{Ht},k_{Ht},z_{Ht})} \\
x_{Mt} &= k_{Mt+1} - (1-\delta_M)k_{Mt} \\
x_{Ht} &= k_{Ht+1} - (1-\delta_H)k_{Ht} \\
x_t &= x_{Mt} + x_{Ht} \\
k_t &= k_{Mt} + k_{Ht} \\
k_{t+1} &= (1-\delta_M)k_{Mt} + (1-\delta_H)k_{Ht} + x_t
\end{aligned}
$$

Plugging in all constraints in the household's maximization problem yields the following unconstrained maximization problem:

$$
\begin{aligned}
\max_{h_{Ht},h_{Mt},k_{Ht},k_{Mt}} U = \sum_{t=0}^{\infty} \beta^t (\frac{b}{e} log[a(w_t(1-\tau_h)h_{Mt} + r_t(1-\tau_k)k_{Mt} + \delta_m \tau_k k_{Mt} + T_t - k_{Mt+1} \\
+(1-\delta_m)k_{Mt} - k_{Ht+1} + (1-\delta_h)k_{Ht})^e + (1-a)(k_{Ht}^{\eta}(z_{Ht}h_{Ht})^{1-\eta})^e)] + (1-b)log(1-h_{Ht}-h_{Mt}))
\end{aligned}
\tag{16}
$$

8.1.1 First order conditions of the household

Taking the derivatives with respect to the choice variables of the household h_{Ht}, h_{Mt}, k_{Ht}, and k_{Mt} result in the first order conditions of the household evaluated at date t.

$$h_{Ht}: \quad \underbrace{(1-a)b(1-\eta)C_t^{-e}c_{Ht}^{e}h_{Ht}^{-1}}_{=u_2(t)g_1(t)} = \underbrace{(1-b)l_t^{-1}}_{=u_3(t)} \tag{17}$$

$$h_{Mt}: \quad \underbrace{ab(1-\tau_h)C_t^{-e}c_{Mt}^{e-1}w_t}_{=u_1(t)(1-\tau_h)w_t} = \underbrace{(1-b)l_t^{-1}}_{=u_4(t)} \tag{18}$$

$$k_{Ht}: \underbrace{a(1-\delta_H)C_t^{-e}c_{Mt}^{e-1}}_{=u_1(t)(1-\delta_H)} + \underbrace{(1-a)\eta C_t^{-e}c_{Ht}^{e}k_{Ht}^{-1}}_{=u_2(t)g_2(t)} = \underbrace{a\beta^{-1}C_{t-1}^{-e}c_{Mt-1}^{e-1}}_{=\beta^{-1}u_1(t-1)} \tag{19}$$

$$k_{Mt}: \underbrace{a(r_t(1-\tau_k)+\delta_M\tau_k+1-\delta_M)C_t^{-e}c_{Mt}^{e-1}}_{=(r_t(1-\tau_k)+\delta_M\tau_k+1-\delta_M)u_1(t)} = \underbrace{a\beta^{-1}C_{t-1}^{-e}c_{Mt-1}^{e-1}}_{=\beta^{-1}u_1(t-1)} \tag{20}$$

Note: All terms below each equation equal the first-order condition expressions in Greenwood et al. (1993).

Equations (17) and (18) are the efficiency conditions concerning the allocation of labor between the market sector and the home sector. Take equation (18) as an example for said efficiency conditions. The RHS of the equation gives you the experienced disutility of a household by allocating one extra unit of labor to market production. Whilst the LHS represents the gain in a household's utility from allocating one extra unit of labor to market production. Intuitively, the benefit comes from a higher market production which will increase a household's utility via higher consumption through higher labor income.

On the other hand equations (19) and (20) are the efficiency conditions for the allocation of capital. Take here equation (19) as an example. Let's assume the household decides at period $t-1$ to purchase one extra unit of household capital at the expense of consuming goods from market production. The disutility of the assumed action is given by the RHS. Whilst the LHS shows the increase in utility in period t. The increase results from higher production in period t thanks to the extra capital unit purchased in period $t-1$ (corresponds to $u_2(t)g_2(t)$) as well as a wealth increase of the household from the extra capital unit purchased in $t-1$ (corresponds to $u_1(t)(1-\delta_h)$). Additionally, both the market and home production budget constraints hold with equality.

8.2 Firm

The firm in this model maximizes its profits by choosing the input factors k_{Mt} and h_{Mt}.

$$\max_{k_{Mt},h_{Mt}} \Pi_t = y_t - r_t k_{Mt} - w_t h_{Mt}$$
$$s.t. \qquad y_t = k_{Mt}^{\theta}(z_{Mt}h_{Mt})^{1-\theta}$$

8.2.1 First order conditions of the firm

The first-order conditions for the firm's problem are stated below.

$$k_{Mt}: \qquad \theta y_t k_{Mt}^{-1} = r_t \tag{21}$$

$$h_{Mt}: (1-\theta) y_t h_{Mt}^{-1} = w_t \tag{22}$$

The firm hires labor for w_t and rents capital for r_t up to the point where marginal products equal factor prices.

8.3 Government

Government expenditures are assumed to be equal to zero, thus the government redistributes all income back to the households via a lump-sum transfer.

$$G_t = w_t \tau_h h_{Mt} + r_t \tau_k k_{Mt} - \delta_M \tau_k k_{Mt} - T_t = 0$$

9 General Equilibrium without Deflating

The competitive equilibrium of this model consists of the utility-maximizing household and profit-maximizing firm. Both agents act on markets such that all markets clear, according to the feasibility condition. In equilibrium, the economy converges to a balanced growth path, where all endogenous variables grow at a constant rate λ. Additionally, feasibility implies that market output is allocated across market consumption, total investment, and government spending, but in our case, government spending equals to zero (see section 4), this results in the resource constraint.

$$y_t = c_{Mt} + x_t \tag{23}$$

The remaining equilibrium conditions consist of the FOCs derived in the previous section of the utility-maximizing household and profit-maximizing firm.

$$(1-a)b(1-\eta)C_t^{-e}c_{Ht}^{e}h_{Ht}^{-1} = (1-b)l_t^{-1} \tag{24}$$

$$ab(1-\tau_h)(1-\theta)C_t^{-e}c_{Mt}^{e-1}y_t h_{Mt}^{-1} = (1-b)l_t^{-1} \tag{25}$$

$$\beta C_t^{-e}[a(1-\delta_H)c_{Mt}^{e-1} + (1-a)\eta c_{Ht}^{e}k_{Ht}^{-1}] = aC_{t-1}^{-e}c_{Mt-1}^{e-1} \tag{26}$$

$$\beta[r_t(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]C_t^{-e}c_{Mt}^{e-1} = C_{t-1}^{-e}c_{Mt-1}^{e-1} \tag{27}$$

$$\theta y_t k_{Mt}^{-1} = r_t \tag{28}$$

$$(1-\theta)y_t h_{Mt}^{-1} = w_t \tag{29}$$

And the remaining constraints of the household, the firm and the government are

$$y_t = k_{Mt}^{\theta}(z_{Mt}h_{Mt})^{1-\theta} \tag{30}$$

$$C_t = (ac_{Mt}^{e} + (1-a)c_{Ht}^{e})^{\frac{1}{e}} \tag{31}$$

$$l_t = 1 - h_{Ht} - h_{Mt} \tag{32}$$

$$c_{Ht} = k_{Ht}^{\eta}(z_{Ht}h_{Ht})^{1-\eta} \tag{33}$$

$$x_{Mt} = k_{Mt+1} - (1-\delta_M)k_{Mt} \tag{34}$$

$$x_{Ht} = k_{Ht+1} - (1-\delta_H)k_{Ht} \tag{35}$$

$$x_t = x_{Mt} + x_{Ht} \tag{36}$$

$$k_t = k_{Mt} + k_{Ht} \tag{37}$$

$$T_t = w_t\tau_h h_{Mt} + r_t\tau_k k_{Mt} - \delta_M\tau_k k_{Mt} \tag{38}$$

The following equations restate the technology shock for the home and market.

$$z_{Mt} = \lambda^t \tilde{z}_{Mt}$$

$$z_{Ht} = \lambda^t \tilde{z}_{Ht}$$

$$log(\tilde{z}_{Mt+1}) = \rho_M log(\tilde{z}_{Mt}) + \epsilon_{Mt+1} \tag{39}$$

$$log(\tilde{z}_{Ht+1}) = \rho_H log(\tilde{z}_{Ht}) + \epsilon_{Ht+1} \tag{40}$$

10 General Equilibrium with Deflating

Since this model assumes a balanced growth path, all endogenous variables grow at the same constant rate λ. Only the time allocations h_{Ht} and h_{Mt} (therefore l_t as well) remain constant. From equation (28), we know, that r_t has to remain constant as well. The deflated endogenous variables are defined below (where $\tilde{v}_t$ denotes the deflated value of variable v_t).

$$\tilde{C}_t = \frac{C_t}{\lambda^t}$$
$$\tilde{c}_{Mt} = \frac{c_{Mt}}{\lambda^t}$$
$$\tilde{c}_{Ht} = \frac{c_{Ht}}{\lambda^t}$$
$$\tilde{k}_t = \frac{k_t}{\lambda^t}$$
$$\tilde{k}_{Mt} = \frac{k_{Mt}}{\lambda^t}$$
$$\tilde{k}_{Ht} = \frac{k_{Ht}}{\lambda^t}$$
$$\tilde{x}_t = \frac{x_t}{\lambda^t}$$
$$\tilde{x}_{Mt} = \frac{x_{Mt}}{\lambda^t}$$
$$\tilde{x}_{Ht} = \frac{x_{Ht}}{\lambda^t}$$
$$\tilde{y}_t = \frac{y_t}{\lambda^t}$$
$$\tilde{T}_t = \frac{T_t}{\lambda^t}$$
$$\tilde{w}_t = \frac{w_t}{\lambda^t}$$
$$\tilde{z}_{Mt} = \frac{z_{Mt}}{\lambda^t}$$
$$\tilde{z}_{Ht} = \frac{z_{Ht}}{\lambda^t}$$

The deflated general equilibrium conditions from the households and firms' maximization problem are stated below. We replace the non-deflated value with the deflated value times the growth rate until time t (e.g., $\tilde{v}_t \lambda^t = v_t$). At first, we deflate the resource constraint and the FOCs of the household and firm:

$$\tilde{y}_t\lambda^t = \tilde{c}_{Mt}\lambda^t + \tilde{x}_t\lambda^t \Rightarrow \tilde{y}_t = \tilde{c}_{Mt} + \tilde{x}_t \tag{41}$$

$$\begin{aligned}(1-a)b(1-\eta)(\tilde{C}_t\lambda^t)^{-e}(\tilde{c}_{Ht}\lambda^t)^e h_{Ht}^{-1} &= (1-b)l_t^{-1} \\ \Rightarrow (1-a)b(1-\eta)\tilde{C}_t^{-e}\tilde{c}_{Ht}^e h_{Ht}^{-1} &= (1-b)l_t^{-1}\end{aligned} \tag{42}$$

$$\begin{aligned}ab(1-\tau_h)(1-\theta)(\tilde{C}_t\lambda^t)^{-e}(\tilde{c}_{Mt}\lambda^t)^{e-1}\tilde{y}_t\lambda^t h_{Mt}^{-1} &= (1-b)l_t^{-1} \\ \Rightarrow ab(1-\tau_h)(1-\theta)\tilde{C}_t^{-e}\tilde{c}_{Mt}^{e-1}\tilde{y}_t h_{Mt}^{-1} &= (1-b)l_t^{-1}\end{aligned} \tag{43}$$

$$\begin{aligned}\beta(\tilde{C}_t\lambda^t)^{-e}[a(1-\delta_H)(\tilde{c}_{Mt}\lambda^t)^{e-1} + (1-a)\eta(\tilde{c}_{Ht}\lambda^t)^e(k_{Ht}\lambda^t)^{-1}] &= a(\tilde{C}_{t-1}\lambda^{t-1})^{-e}(\tilde{c}_{Mt-1}\lambda^{t-1})^{e-1} \\ \Rightarrow \beta\tilde{C}_t^{-e}[a(1-\delta_H)\tilde{c}_{Mt}^{e-1} + (1-a)\eta\tilde{c}_{Ht}^e k_{Ht}^{-1}] &= a\lambda\tilde{C}_{t-1}^{-e}\tilde{c}_{Mt-1}^{e-1}\end{aligned} \tag{44}$$

$$\begin{aligned}\beta[r_t(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M](\tilde{C}_t\lambda^t)^{-e}(\tilde{c}_{Mt}\lambda^t)^{e-1} &= (\tilde{C}_{t-1}\lambda^{t-1})^{-e}(\tilde{c}_{Mt-1}\lambda^{t-1})^{e-1} \\ \Rightarrow \beta[r_t(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\tilde{C}_t^{-e}\tilde{c}_{Mt}^{e-1} &= \lambda\tilde{C}_{t-1}^{-e}\tilde{c}_{Mt-1}^{e-1}\end{aligned} \tag{45}$$

$$\theta\tilde{y}_t\lambda^t(\tilde{k}_{Mt}\lambda^t)^{-1} = r_t \Rightarrow \theta\tilde{y}_t\tilde{k}_{Mt}^{-1} = r_t \tag{46}$$

$$(1-\theta)\tilde{y}_t\lambda^t h_{Mt}^{-1} = \tilde{w}_t\lambda^t \Rightarrow (1-\theta)\tilde{y}_t h_{Mt}^{-1} = \tilde{w}_t \tag{47}$$

And then the remaining constraints:

$$\tilde{y}_t\lambda^t = (\tilde{k}_{Mt}\lambda^t)^\theta(\tilde{z}_{Mt}\lambda^t h_{Mt})^{1-\theta} \Rightarrow \tilde{y}_t = \tilde{k}_{Mt}^\theta(\tilde{z}_{Mt}h_{Mt})^{1-\theta} \tag{48}$$

$$\tilde{C}_t\lambda^t = (a(\tilde{c}_{Mt}\lambda^t)^e + (1-a)(\tilde{c}_{Ht}\lambda^t)^e)^{\frac{1}{e}} \Rightarrow \tilde{C}_t = (a\tilde{c}_{Mt}^e + (1-a)\tilde{c}_{Ht}^e)^{\frac{1}{e}} \tag{49}$$

$$l_t = 1 - h_{Ht} - h_{Mt} \tag{50}$$

$$\tilde{c}_{Ht}\lambda^t = (\tilde{k}_{Ht}\lambda^t)^\eta(\tilde{z}_{Ht}\lambda^t h_{Ht})^{1-\eta} \Rightarrow \tilde{c}_{Ht} = \tilde{k}_{Ht}^\eta(\tilde{z}_{Ht}h_{Ht})^{1-\eta} \tag{51}$$

$$\tilde{x}_{Mt}\lambda^t = \tilde{k}_{Mt+1}\lambda^{t+1} - (1-\delta_M)\tilde{k}_{Mt}\lambda^t \Rightarrow \tilde{x}_{Mt} = \lambda\tilde{k}_{Mt+1} - (1-\delta_M)\tilde{k}_{Mt} \tag{52}$$

$$\tilde{x}_{Ht}\lambda^t = \tilde{k}_{Ht+1}\lambda^{t+1} - (1-\delta_H)\tilde{k}_{Ht}\lambda^t \Rightarrow \tilde{x}_{Ht} = \lambda\tilde{k}_{Ht+1} - (1-\delta_H)\tilde{k}_{Ht} \tag{53}$$

$$\tilde{x}_t\lambda^t = \tilde{x}_{Mt}\lambda^t + \tilde{x}_{Ht}\lambda^t \Rightarrow \tilde{x}_t = \tilde{x}_{Mt} + \tilde{x}_{Ht} \tag{54}$$

$$\tilde{k}_t\lambda^t = \tilde{k}_{Mt}\lambda^t + \tilde{k}_{Ht}\lambda^t \Rightarrow \tilde{k}_t = \tilde{k}_{Mt} + \tilde{k}_{Ht} \tag{55}$$

$$\begin{aligned}\tilde{T}_t\lambda^t &= \tilde{w}_t\lambda^t\tau_h h_{Mt} + r_t\tau_k\tilde{k}_{Mt}\lambda^t - \delta_M\tau_k\tilde{k}_{Mt}\lambda^t \\ \Rightarrow \tilde{T}_t &= \tilde{w}_t\tau_h h_{Mt} + r_t\tau_k\tilde{k}_{Mt} - \delta_M\tau_k\tilde{k}_{Mt}\end{aligned} \tag{56}$$

$$log(\tilde{z}_{Mt+1}) = \rho_M log(\tilde{z}_{Mt}) + \epsilon_{Mt+1} \tag{57}$$

$$log(\tilde{z}_{Ht+1}) = \rho_H log(\tilde{z}_{Ht}) + \epsilon_{Ht+1} \tag{58}$$

The previously stated equations constitute the deflated general equilibrium of the model, where the variables are denoted in levels.

11 Steady State

For the steady state calculation, we assume h_H, h_M, and r to be fixed. On the contrary, the parameter values a, b, and β are calculated based on the resulting steady state values of specific variables. Below are the equations from the deflated general equilibrium without the time indices.

$$\tilde{y} = \tilde{c}_M + \tilde{x} \tag{59}$$

$$(1-a)b(1-\eta)\tilde{C}^{-e}\tilde{c}_H^e h_H^{-1} = (1-b)l^{-1} \tag{60}$$

$$ab(1-\tau_h)(1-\theta)\tilde{C}^{-e}\tilde{c}_M^{e-1}\tilde{y}h_M^{-1} = (1-b)l^{-1} \tag{61}$$

$$\beta\tilde{C}^{-e}[a(1-\delta_H)\tilde{c}_M^{e-1} + (1-a)\eta\tilde{c}_H^e\tilde{k}_H^{-1}] = a\lambda\tilde{C}^{-e}\tilde{c}_M^{e-1}$$

$$\Rightarrow (1-a)\eta\tilde{c}_H^e k_H^{-1} = a\tilde{c}_M^{e-1}(\lambda\beta^{-1} - 1 + \delta_H) \tag{62}$$

$$\beta[r(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\tilde{C}^{-e}\tilde{c}_M^{e-1} = \lambda\tilde{C}^{-e}\tilde{c}_M^{e-1}$$

$$\Rightarrow r(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M = \lambda\beta^{-1} \tag{63}$$

$$\theta\tilde{y}\tilde{k}_M^{-1} = r \tag{64}$$

$$(1-\theta)\tilde{y}h_M^{-1} = \tilde{w} \tag{65}$$

$$\tilde{y} = \tilde{k}_M^{\theta}(\tilde{z}_M h_M)^{1-\theta} \tag{66}$$

$$\tilde{C} = (a\tilde{c}_M^e + (1-a)\tilde{c}_H^e)^{\frac{1}{e}} \tag{67}$$

$$l = 1 - h_H - h_M \tag{68}$$

$$\tilde{c}_H = \tilde{k}_H^{\eta}(\tilde{z}_H h_H)^{1-\eta} \tag{69}$$

$$\tilde{x}_M = \lambda\tilde{k}_M - (1-\delta_M)\tilde{k}_M \Rightarrow \tilde{x}_M = (\lambda - 1 + \delta_M)\tilde{k}_M \tag{70}$$

$$\tilde{x}_H = \lambda\tilde{k}_H - (1-\delta_H)\tilde{k}_H \Rightarrow \tilde{x}_{Ht} = (\lambda - 1 + \delta_H)\tilde{k}_H \tag{71}$$

$$\tilde{x} = \tilde{x}_M + \tilde{x}_H \tag{72}$$

$$\tilde{k} = \tilde{k}_M + \tilde{k}_H \tag{73}$$

$$\tilde{T} = \tilde{w}\tau_h h_M + r\tau_k\tilde{k}_M - \delta_M\tau_k\tilde{k}_M \tag{74}$$

$$log(\tilde{z}_M) = \rho_M log(\tilde{z}_M) \tag{75}$$

$$log(\tilde{z}_H) = \rho_H log(\tilde{z}_H) \tag{76}$$

From the last two equations, we can derive $\tilde{z}_M^* = \tilde{z}_H^* = 1$ (note, that $\epsilon_M = \epsilon_H = 0$ in steady state). In the next step, we determine the r^* based on the FOC of the household w.r.t. k_M

$$r^* = \frac{\lambda\beta^{-1} - 1 + \delta_M - \delta_M\tau_k}{1 - \tau_k} \tag{77}$$

Use the firms's FOC w.r.t. k_M to get

$$\frac{\tilde{k}_M}{\tilde{y}} = \frac{\theta}{r} \tag{78}$$

Divide the production function by $\tilde{k}_M$ and rearrange to get

$$\frac{\tilde{k}_M}{h_M} = \tilde{z}_M \left(\frac{\tilde{k}_M}{\tilde{y}}\right)^{\frac{1}{1-\theta}} \tag{79}$$

$\tilde{k}_M/\tilde{y}$ and $\tilde{k}_M/h_M$ are already defined, hence from the firm's FOC w.r.t. h_M

$$\tilde{w}^* = (1-\theta)\frac{\tilde{y}}{h_M} = (1-\theta)\left(\frac{\tilde{k}_M}{\tilde{y}}\right)^{-1}\frac{\tilde{k}_M}{h_M} \tag{80}$$

h_M and k_M/h_M are defined already, hence from the production function

$$\tilde{y}^* = \left(\frac{\tilde{k}_M}{h_M}\right)^{\theta} h_M \tilde{z}_M^{1-\theta} \tag{81}$$

Combine the household's FOC w.r.t. h_M and h_H

$$\frac{(1-a)(1-\eta)\tilde{c}_H^e h_H^{-1}}{a(1-\tau_h)\tilde{c}_M^{e-1}\tilde{w}^*} = 1 \Rightarrow \frac{\tilde{c}_H^e}{\tilde{c}_M^{e-1}} = \frac{a(1-\tau_h)\tilde{w}^* h_H}{(1-a)(1-\eta)} \tag{82}$$

Rearrange the FOC of the household w.r.t. k_H and plug in the previous expression

$$\tilde{k}_H^* = \frac{\tilde{c}_H^e}{\tilde{c}_M^{e-1}}\frac{(1-a)\eta}{a(\lambda\beta^{-1}-1+\delta_H)} = \frac{\eta(1-\tau_h)\tilde{w}^* h_H}{(1-\eta)(\lambda\beta^{-1}-1+\delta_H)} \tag{83}$$

The remaining steady states of the endogenous variables can be obtained by plugging in the previous results.

$$\tilde{k}_M^* = \frac{\tilde{k}_M}{h_M} h_M \tag{84}$$

$$\tilde{k}^* = \tilde{k}_H^* + \tilde{k}_M^* \tag{85}$$

$$\tilde{x}_M^* = (\lambda - 1 + \delta_M)\tilde{k}_M^* \tag{86}$$

$$\tilde{x}_H^* = (\lambda - 1 + \delta_M)\tilde{k}_H^* \tag{87}$$

$$\tilde{x}^* = \tilde{x}_H^* + \tilde{x}_M^* \tag{88}$$

$$\tilde{c}_M^* = \tilde{y}^* + \tilde{x}_M^* + \tilde{x}_H^* \tag{89}$$

$$\tilde{c}_H^* = \tilde{k}_H^{*\eta}(\tilde{z}_H^* h_H)^{1-\eta} \tag{90}$$

$$\tilde{T}^* = \tilde{w}^*\tau_h h_M + r^*\tau_k \tilde{k}_M^* - \delta_M \tau_k \tilde{k}_M^* \tag{91}$$

To determine the parameter a, we solve the household FOC w.r.t. k_H for a

$$a = [\eta^{-1}\tilde{c}_M^{*e-1}\tilde{c}_H^{*-e}\tilde{k}_H^*(\lambda\beta^{-1} - 1 + \delta_H) + 1]^{-1} \tag{92}$$

After determining the value of a, we can derive the steady state of the total consumption.

$$\tilde{C}^{*} = (a(\tilde{c}_M^{*})^e + (1-a)(\tilde{c}_H^{*})^e)^{\frac{1}{e}} \tag{93}$$

Similar to the parameter a, we solve the household FOC w.r.t. h_H for b

$$b = [(1-a)(1-\eta)\tilde{C}^{*-e}\tilde{c}_H^{*e}h_H^{-1}l + 1]^{-1} \tag{94}$$

12 Log-Transformation

In this section, we apply the log-transformation to every variable in the deflated model for a more intuitive interpretation of the results. Hence, we make use of the fact that $\hat{v}_t = log(\tilde{v}_t) \Rightarrow \exp(\hat{v}_t) = \tilde{v}_t$ and exchange in our model every variable $\tilde{v}_t$ with $\exp(\hat{v}_t)$. Firstly, we apply the transformation to the resource constraint.

$$\exp(\hat{y}_t) = \exp(\hat{c}_{Mt}) + \exp(\hat{x}_t) \tag{95}$$

Secondly, to the FOCs of the household and the firm

$$\begin{aligned}
(1-a)b(1-\eta)\exp(\hat{C}_t)^{-e}\exp(\hat{c}_{Ht})^e\exp(\hat{h}_{Ht})^{-1} &= (1-b)\exp(\hat{l}_t)^{-1} \\
\Rightarrow (1-a)b(1-\eta)\exp(-\hat{C}_t e)\exp(\hat{c}_{Ht}e)\exp(-\hat{h}_{Ht}) &= (1-b)\exp(-l_t) \\
\Rightarrow (1-a)b(1-\eta)\exp(-\hat{C}_t e + \hat{c}_{Ht}e - \hat{h}_{Ht}) & \\
&= (1-b)\exp(-l_t) \qquad (96) \\
ab(1-\tau_h)(1-\theta)\exp(\hat{C}_t)^{-e}\exp(\hat{c}_{Mt})^{e-1}\exp(\hat{y}_t)\exp(\hat{h}_{Mt})^{-1} &= (1-b)\exp(\hat{l}_t)^{-1} \\
\Rightarrow ab(1-\tau_h)(1-\theta)\exp(-\hat{C}_t e)\exp(\hat{c}_{Mt}(e-1))\exp(\hat{y}_t)\exp(-\hat{h}_{Mt}) &= (1-b)\exp(-\hat{l}_t) \\
\Rightarrow ab(1-\tau_h)(1-\theta)\exp(-\hat{C}_t e + \hat{c}_{Mt}(e-1) + \hat{y}_t - \hat{h}_{Mt}) & \\
&= (1-b)\exp(-\hat{l}_t) \qquad (97) \\
\beta\exp(\hat{C}_t)^{-e}[a(1-\delta_H)\exp(\hat{c}_{Mt})^{e-1} + (1-a)\eta\exp(\hat{c}_{Ht})^e\exp(\hat{k}_{Ht})^{-1}] &= a\lambda\exp(\hat{C}_{t-1})^{-e}\exp(\hat{c}_{Mt-1})^{e-1} \\
\Rightarrow \beta\exp(-\hat{C}_t e)[a(1-\delta_H)\exp(\hat{c}_{Mt}(e-1)) + (1-a)\eta\exp(\hat{c}_{Ht}e - \hat{k}_{Ht})] & \\
&= a\lambda\exp(-\hat{C}_{t-1}e + \hat{c}_{Mt-1}(e-1)) \qquad (98) \\
\beta[\exp(\hat{r}_t)(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\exp(\hat{C}_t)^{-e}\exp(\hat{c}_{Mt})^{e-1} &= \lambda\exp(\hat{C}_{t-1})^{-e}\exp(\hat{c}_{Mt-1})^{e-1} \\
\Rightarrow \beta[\exp(\hat{r}_t)(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\exp(-\hat{C}_t e + \hat{c}_{Mt}(e-1)) & \\
&= \lambda\exp(-\hat{C}_{t-1}e + \hat{c}_{Mt-1}(e-1)) \qquad (99)
\end{aligned}$$

And thirdly, to the remaining constraints in the model.

$$\theta \exp(\hat{y}_t) \exp(\hat{k}_{Mt})^{-1} = \exp(\hat{r}_t) \Rightarrow \theta \exp(\hat{y}_t - \hat{k}_{Mt}) = \exp(\hat{r}_t) \tag{100}$$

$$(1-\theta) \exp(\hat{y}_t) \exp(\hat{h}_{Mt})^{-1} = \exp(\hat{w}_t) \Rightarrow (1-\theta) \exp(\hat{y}_t - \hat{h}_{Mt}) = \exp(\hat{w}_t) \tag{101}$$

$$\begin{aligned} \exp(\hat{y}_t) &= \exp(\hat{k}_{Mt})^{\theta} (\exp(\hat{z}_{Mt}) \exp(\hat{h}_{Mt}))^{1-\theta} \\ \Rightarrow \exp(\hat{y}_t) &= \exp(\theta \hat{k}_{Mt} + (1-\theta)(\hat{z}_{Mt} + \hat{h}_{Mt})) \end{aligned} \tag{102}$$

$$\begin{aligned} \exp(\hat{C}_t) &= (a \exp(\hat{c}_{Mt})^e + (1-a) \exp(\hat{c}_{Ht})^e)^{\frac{1}{e}} \\ \Rightarrow \exp(\hat{C}_t) &= (a \exp(\hat{c}_{Mt} e) + (1-a) \exp(\hat{c}_{Ht} e))^{\frac{1}{e}} \end{aligned} \tag{103}$$

$$\exp(\hat{l}_t) = 1 - \exp(\hat{h}_{Ht}) - \exp(\hat{h}_{Mt}) \tag{104}$$

$$\begin{aligned} \exp(\hat{c}_{Ht}) &= \exp(\hat{k}_{Ht})^{\eta} (\exp(\hat{z}_{Ht}) \exp(\hat{h}_{Ht}))^{1-\eta} \\ \Rightarrow \exp(\hat{c}_{Ht}) &= \exp(\eta \hat{k}_{Ht} + (1-\eta)(\hat{z}_{Ht} + \hat{h}_{Ht})) \end{aligned} \tag{105}$$

$$\exp(\hat{x}_{Mt}) = \lambda \exp(\hat{k}_{Mt+1}) - (1-\delta_M) \exp(\hat{k}_{Mt}) \tag{106}$$

$$\exp(\hat{x}_{Ht}) = \lambda \exp(\hat{k}_{Ht+1}) - (1-\delta_H) \exp(\hat{k}_{Ht}) \tag{107}$$

$$\exp(\hat{x}_t) = \exp(\hat{x}_{Mt}) + \exp(\hat{x}_{Ht}) \tag{108}$$

$$\exp(\hat{k}_t) = \exp(\hat{k}_{Mt}) + \exp(\hat{k}_{Ht}) \tag{109}$$

$$\begin{aligned} \exp(\hat{T}_t) &= \exp(\hat{w}_t)\tau_h \exp(\hat{h}_{Mt}) + \exp(\hat{r}_t)\tau_k \exp(\hat{k}_{Mt}) \\ &\quad - \delta_M \tau_k \exp(\hat{k}_{Mt}) \\ \Rightarrow \exp(\hat{T}_t) &= \exp(\hat{w}_t + \hat{h}_{Mt})\tau_h + \exp(\hat{r}_t + \hat{k}_{Mt})\tau_k \\ &\quad - \delta_M \tau_k \exp(\hat{k}_{Mt}) \end{aligned} \tag{110}$$

$$\hat{z}_{Mt+1} = \rho_M \hat{z}_{Mt} + \epsilon_{Mt+1} \tag{111}$$

$$\hat{z}_{Ht+1} = \rho_H \hat{z}_{Ht} + \epsilon_{Ht+1} \tag{112}$$

13 Impulse Response Functions

Dynare provides us with the Impulse Response Functions (IRF) of $\log(\tilde{y}_t)$. However, only the IRF of $\log(y_t)$ is of interest. The IRFs are with respect to ϵ_M and ϵ_H, but to simplify the next derivation, we use ϵ.

$$\frac{\partial \log(y_t)}{\partial \epsilon_t} = \frac{\partial \log(\tilde{y}_t \lambda^t)}{\partial \epsilon_t} = \frac{\partial \log(\tilde{y}_t)}{\partial \epsilon_t} + \underbrace{\frac{\partial \log(\lambda^t)}{\partial \epsilon_t}}_{=0} = \frac{\partial \log(\tilde{y}_t)}{\partial \epsilon_t} = \frac{\partial \hat{y}_t}{\partial \epsilon_t}$$

Based on the assumption, that the growth rate λ is constant, we can directly use the Dynare output of the IRFs for $\hat{y}_t = \log(\tilde{y}_t)$.

14 Summary

In this section, we list all the equations as used in the Dynare code for the deflated log-transformed model. The variables and parameters remain as listed in section 7.

$$\exp(\hat{c}_{Mt}) + \exp(\hat{x}_t) = \exp(\hat{y}_t) \tag{113}$$

$$\begin{aligned}(1-a)b(1-\eta)\exp(-\hat{C}_t e + \hat{c}_{Ht} e - \hat{h}_{Ht}) \\ = (1-b)\exp(-l_t)\end{aligned} \tag{114}$$

$$\begin{aligned}ab(1-\tau_h)(1-\theta)\exp(-\hat{C}_t e + \hat{c}_{Mt}(e-1) + \hat{y}_t - \hat{h}_{Mt}) \\ = (1-b)\exp(-\hat{l}_t)\end{aligned} \tag{115}$$

$$\begin{aligned}\beta\exp(-\hat{C}_t e)[a(1-\delta_H)\exp(\hat{c}_{Mt}(e-1)) + (1-a)\eta\exp(\hat{c}_{Ht} e - \hat{k}_{Ht})] \\ = a\lambda\exp(-\hat{C}_{t-1} e + \hat{c}_{Mt-1}(e-1))\end{aligned} \tag{116}$$

$$\begin{aligned}\beta[\exp(\hat{r}_t)(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\exp(-\hat{C}_t e + \hat{c}_{Mt}(e-1)) \\ = \lambda\exp(-\hat{C}_{t-1} e + \hat{c}_{Mt-1}(e-1))\end{aligned} \tag{117}$$

$$\theta\exp(\hat{y}_t - \hat{k}_{Mt}) = \exp(\hat{r}_t) \tag{118}$$

$$(1-\theta)\exp(\hat{y}_t - \hat{h}_{Mt}) = \exp(\hat{w}_t) \tag{119}$$

$$\exp(\theta\hat{k}_{Mt} + (1-\theta)(\hat{z}_{Mt} + \hat{h}_{Mt})) = \exp(\hat{y}_t) \tag{120}$$

$$(a\exp(\hat{c}_{Mt} e) + (1-a)\exp(\hat{c}_{Ht} e))^{\frac{1}{e}} = \exp(\hat{C}_t) \tag{121}$$

$$1 - \exp(\hat{h}_{Ht}) - \exp(\hat{h}_{Mt}) = \exp(\hat{l}_t) \tag{122}$$

$$\exp(\eta\hat{k}_{Ht} + (1-\eta)(\hat{z}_{Ht} + \hat{h}_{Ht})) = \exp(\hat{c}_{Ht}) \tag{123}$$

$$\lambda\exp(\hat{k}_{Mt+1}) - (1-\delta_M)\exp(\hat{k}_{Mt}) = \exp(\hat{x}_{Mt}) \tag{124}$$

$$\lambda\exp(\hat{k}_{Ht+1}) - (1-\delta_H)\exp(\hat{k}_{Ht}) = \exp(\hat{x}_{Ht}) \tag{125}$$

$$\exp(\hat{x}_{Mt}) + \exp(\hat{x}_{Ht}) = \exp(\hat{x}_t) \tag{126}$$

$$\exp(\hat{k}_{Mt}) + \exp(\hat{k}_{Ht}) = \exp(\hat{k}_t) \tag{127}$$

$$\exp(\hat{w}_t + \hat{h}_{Mt})\tau_h + \exp(\hat{r}_t + \hat{k}_{Mt})\tau_k - \delta_M\tau_k\exp(\hat{k}_{Mt}) = \exp(\hat{T}_t) \tag{128}$$

$$\rho_M\hat{z}_{Mt} + \epsilon_{Mt+1} = \hat{z}_{Mt+1} \tag{129}$$

$$\rho_H\hat{z}_{Ht} + \epsilon_{Ht+1} = \hat{z}_{Ht+1} \tag{130}$$

15 Extensions

15.1 Minimal Role of Home Production

In the first extension to their model, Greenwood et al. (1993) set the parameter $e = 0$, which implies the elasticity between c_{Mt} and c_{Ht} to be unity. This model aims to minimize the role of home production and according to Greenwood et al. (1993) produces results similar to a model without home production. This model configuration is also used in Greenwood and Hercowitz (1991) or Greenwood et al. (2020).

By setting $e = 0$, the function for total consumption (3) reduces to a Cobb-Douglas function (i.e. $C_t = c_{Mt}^a c_{Ht}^{1-a}$), this yields a simplified instantaneous utility function V:

$$V = ab\log(c_{Mt}) + (1-a)b\log(c_{Ht}) + (1-b)\log(1 - h_{Mt} - h_{Ht}) \tag{131}$$

By plugging the market budget and home production constraint into the objective function V, we derive the FOCs of the lifetime utility of the household.

$$\begin{aligned}
h_{Ht}: \quad & (1-a)b(1-\eta)h_{Ht}^{-1} &&= (1-b)l_t^{-1} && (132)\\
h_{Mt}: \quad & ab(1-\tau_h)c_{Mt}^{-1}w_t &&= (1-b)l_t^{-1} && (133)\\
k_{Ht}: \quad & a(1-\delta_H)c_{Mt}^{-1} + (1-a)\eta k_{Ht}^{-1} &&= a\beta^{-1}c_{Mt-1}^{-1} && (134)\\
k_{Mt}: \quad & a(r_t(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M)c_{Mt}^{-1} &&= a\beta^{-1}c_{Mt-1}^{-1} && (135)
\end{aligned}$$

Since the firm's profit maximization problem is not changed, the FOCs of the firm are as stated in the previous sections. In the next step, we deflate the updated FOCs and the Cobb-Douglas function for the total consumption as in section 10:

$$\begin{aligned}
(1-a)b(1-\eta)h_{Ht}^{-1} &= (1-b)l_t^{-1} && (136)\\
ab(1-\tau_h)(\lambda^t\tilde{c}_{Mt})^{-1}\lambda^t\tilde{w}_t &= (1-b)l_t^{-1}\\
\Rightarrow ab(1-\tau_h)\tilde{c_{Mt}}^{-1}\tilde{w}_t &= (1-b)l_t^{-1} && (137)\\
a(1-\delta_H)(\lambda^t\tilde{c}_{Mt})^{-1} + (1-a)\eta(\lambda^t\tilde{k}_{Ht})^{-1} &= \beta^{-1}(\lambda^{t-1}\tilde{c}_{Mt-1})^{-1}\\
\Rightarrow a(1-\delta_H)\tilde{c}_{Mt}^{-1} + (1-a)\eta\tilde{k}_{Ht}^{-1} &= a\beta^{-1}\lambda\tilde{c}_{Mt-1}^{-1} && (138)\\
(r_t(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M)(\lambda^t\tilde{c}_{Mt})^{-1} &= \beta^{-1}(\lambda^{t-1}\tilde{c}_{Mt-1})^{-1}\\
\Rightarrow (r_t(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M)\tilde{c}_{Mt}^{-1} &= \beta^{-1}\lambda\tilde{c}_{Mt-1}^{-1} && (139)\\
\lambda^t\tilde{C}_t = (\lambda^t\tilde{c}_{Mt})^a(\lambda^t\tilde{c}_{Ht})^{1-a} &\Rightarrow \tilde{C}_t = \tilde{c}_{Mt}^a\tilde{c}_{Ht}^{1-a} && (140)
\end{aligned}$$

In the final step, we log-transform the updated FOCs the new expression for the total consumption as in section 12.

$$
\begin{aligned}
(1-a)b(1-\eta)\exp(\hat{h}_{Ht})^{-1} &= (1-b)\exp(\hat{l}_t)^{-1} \\
\Rightarrow (1-a)b(1-\eta)\exp(-\hat{h}_{Ht}) &= (1-b)\exp(-\hat{l}_t) \qquad (141) \\
ab(1-\tau_h)\exp(\hat{c}_{Mt})^{-1}\exp(\hat{w}_t) &= (1-b)\exp(\hat{l}_t)^{-1} \\
\Rightarrow ab(1-\tau_h)\exp(-\hat{c}_{Mt}+\hat{w}_t) &= (1-b)\exp(-\hat{l}_t) \qquad (142) \\
a(1-\delta_H)\exp(\hat{c}_{Mt})^{-1} + (1-a)\eta\exp(\hat{k}_{Ht})^{-1} &= a\beta^{-1}\lambda\exp(\hat{c}_{Mt-1})^{-1} \\
\Rightarrow a(1-\delta_H)\exp(-\hat{c}_{Mt}) + (1-a)\eta\exp(-\hat{k}_{Ht}) &= a\beta^{-1}\lambda\exp(-\hat{c}_{Mt-1}) \qquad (143) \\
[\exp(\hat{r}_t)(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\exp(\hat{c}_{Mt})^{-1} &= \beta^{-1}\lambda\exp(\hat{c}_{Mt-1})^{-1} \\
\Rightarrow [\exp(\hat{r}_t)(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\exp(-\hat{c}_{Mt}) &= \beta^{-1}\lambda\exp(-\hat{c}_{Mt-1}) \qquad (144) \\
\exp(\hat{C}_t) &= \exp(\hat{c}_{Mt})^a\exp(\hat{c}_{Ht})^{1-a} \\
\Rightarrow \exp(\hat{C}_t) &= \exp(a\hat{c}_{Mt} + (1-a)\hat{c}_{Ht}) \qquad (145)
\end{aligned}
$$

15.1.1 Steady State

We can recover the steady state, as we did in section 11, by simply setting $e = 0$. Just the calculation of the steady state total consumption in equation (93) changes to

$$C^* = (c_M^*)^a(c_H^*)^{1-a}. \qquad (146)$$

15.1.2 Summary

We just have to exchange equation (121) with (145), therefore we refrain from listing all equations again. The other equations are recovered by setting $e = 0$.

15.2 General Home Production Function

In an earlier paper Greenwood and Hercowitz (1991) assume a more general home production function, which has not been used up to this point of the model. The more general home production function is given by:

$$c_{Ht} = g(h_{Ht}, k_{Ht}, z_{Ht}) = [\eta k_{Ht}^{\Psi} + (1-\eta)(z_{Ht}h_{Ht})^{\Psi}]^{\frac{1}{\Psi}} \qquad (147)$$

It's also assumed that the technology shocks z_H and z_M are highly correlated such that once a shock hits the market it highly influences the home as well. This allows for the case when a positive shock arrives to shift out hours from home production to market production whilst at home the working hours increase in efficiency as well. The FOCs of the household w.r.t. h_{Ht} and k_{Ht} are updated as follows:

$$h_{Ht}: \quad \underbrace{(1-a)b(1-\eta)C_t^{-e}c_{Ht}^{e-\Psi}z_{Ht}^{\Psi}h_{Ht}^{\Psi-1}}_{=u_2(t)g_1(t)} = \underbrace{(1-b)l_t^{-1}}_{=u_3(t)} \tag{148}$$

$$k_{Ht}: \underbrace{a(1-\delta_H)C_t^{-e}c_{Mt}^{e-1}}_{=u_1(t)(1-\delta_H)} + \underbrace{(1-a)\eta C_t^{-e}c_{Ht}^{e-\Psi}k_{Ht}^{\Psi-1}}_{=u_2(t)g_2(t)} = \underbrace{a\beta^{-1}C_{t-1}^{-e}c_{Mt-1}^{e-1}}_{=\beta^{-1}u_1(t-1)} \tag{149}$$

The remaining FOCs of the household and the firm do not change with this general home production function. In the next step, we deflate the updated FOCs and general home production as in section 10:

$$(1-a)b(1-\eta)(\lambda^t\tilde{C}_t)^{-e}(\lambda^t\tilde{c}_{Ht})^{e-\Psi}(\lambda^t\tilde{z}_{Ht})^{\Psi}h_{Ht}^{\Psi-1} = (1-b)l_t^{-1}$$

$$\Rightarrow (1-a)b(1-\eta)\tilde{C}_t^{-e}\tilde{c}_{Ht}^{e-\Psi}\tilde{z}_{Ht}^{\Psi}h_{Ht}^{\Psi-1} = (1-b)l_t^{-1} \tag{150}$$

$$\beta(\lambda^t\tilde{C}_t)^{-e}[a(1-\delta_H)(\lambda^t\tilde{c}_{Mt})^{e-1} + (1-a)\eta(\lambda^t\tilde{c}_{Ht})^{e-\Psi}(\lambda^t\tilde{k}_{Ht})^{\Psi-1}] = a(\lambda^{t-1}\tilde{C}_{t-1})^{-e}(\lambda^{t-1}\tilde{c}_{Mt-1})^{e-1}$$

$$\Rightarrow \beta\tilde{C}_t^{-e}[a(1-\delta_H)\tilde{c}_{Mt}^{e-1} + (1-a)\eta\tilde{c}_{Ht}^{e-\Psi}\tilde{k}_{Ht}^{\Psi-1}] = a\lambda\tilde{C}_{t-1}^{-e}\tilde{c}_{Mt-1}^{e-1} \tag{151}$$

$$\lambda^t\tilde{c}_{Ht} = [\eta(\lambda^t\tilde{k}_{Ht})^{\Psi} + (1-\eta)(\lambda^t\tilde{z}_{Ht}h_{Ht})^{\Psi}]^{\frac{1}{\Psi}}$$

$$\Rightarrow \tilde{c}_{Ht} = [\eta\tilde{k}_{Ht}^{\Psi} + (1-\eta)(\tilde{z}_{Ht}h_{Ht})^{\Psi}]^{\frac{1}{\Psi}} \tag{152}$$

In the final step, we log-transform the updated FOCs as in section 12.

$$(1-a)b(1-\eta)\exp(\hat{C}_t)^{-e}\exp(\hat{c}_{Ht})^{e-\Psi}\exp(\hat{z}_{Ht})^{\Psi}\exp(\hat{h}_{Ht})^{\Psi-1}$$
$$= (1-b)\exp(\hat{l}_t)^{-1}$$
$$\Rightarrow (1-a)b(1-\eta)\exp[\hat{c}_{Ht}(e-\Psi) + \hat{z}_{Ht}\Psi + \hat{h}_{Ht}(\Psi-1) - \hat{C}_te]$$
$$= (1-b)\exp(-\hat{l}_t) \tag{153}$$

$$\beta\exp(\hat{C}_t)^{-e}[a(1-\delta_H)\exp(\hat{c}_{Mt})^{e-1} + (1-a)\eta\exp(\hat{c}_{Ht})^{e-\Psi}\exp(\hat{k}_{Ht})^{\Psi-1}]$$
$$= a\lambda\exp(\hat{C}_{t-1})^{-e}\exp(\hat{c}_{Mt-1})^{e-1}$$
$$\Rightarrow \beta\exp(-\hat{C}_te)\{a(1-\delta_H)\exp(\hat{c}_{Mt}(e-1)) + (1-a)\eta\exp[\hat{c}_{Ht}(e-\Psi) + \hat{k}_{Ht}(\Psi-1)]\}$$
$$= a\lambda\exp(-\hat{C}_{t-1}e + \hat{c}_{Mt-1}(e-1)) \tag{154}$$

$$\exp(\hat{c}_{Ht}) = [\eta\exp(\hat{k}_{Ht})^{\Psi} + (1-\eta)(\exp(\hat{z}_{Ht})\exp(\hat{h}_{Ht}))^{\Psi}]^{\frac{1}{\Psi}}$$
$$\Rightarrow \exp(\hat{c}_{Ht}) = [\eta\exp(\hat{k}_{Ht}\Psi) + (1-\eta)\exp((\hat{z}_{Ht} + \hat{h}_{Ht})\Psi)]^{\frac{1}{\Psi}} \tag{155}$$

15.2.1 Steady State

To calculate the steady state, we can simplify equations (150) - (152) as stated below:

$$(1-a)b(1-\eta)\tilde{C}^{-e}\tilde{c}_H^{e-\Psi}\tilde{z}_H^{\Psi}h_H^{\Psi-1} = (1-b)l^{-1} \tag{156}$$

$$\beta\tilde{C}^{-e}[a(1-\delta_H)\tilde{c}_M^{e-1} + (1-a)\eta\tilde{c}_H^{e-\Psi}\tilde{k}_H^{\Psi-1}] = a\lambda\tilde{C}^{-e}\tilde{c}_M^{e-1}$$

$$\Rightarrow (1-a)\eta\tilde{c}_H^{e-\Psi}\tilde{k}_H^{\Psi-1} = a\tilde{c}_M^{e-1}(\lambda\beta^{-1} - 1 + \delta_H) \tag{157}$$

$$[\eta\tilde{k}_H^{\Psi} + (1-\eta)(\tilde{z}_H h_H)^{\Psi}]^{\frac{1}{\Psi}} = \tilde{c}_H \tag{158}$$

To determine the steady state, we need to replace equation (82) with the following expression:

$$\frac{(1-a)(1-\eta)\tilde{c}_H^{e-\Psi}h_H^{\Psi-1}\tilde{z}_H^{*\Psi}}{a(1-\tau_h)\tilde{c}_M^{e-1}\tilde{w}^*} = 1 \Rightarrow \frac{\tilde{c}_H^{e-\Psi}}{\tilde{c}_M^{e-1}} = \frac{a(1-\tau_h)\tilde{w}^*}{(1-a)(1-\eta)h_H^{\Psi-1}\tilde{z}_H^{*\Psi}} \tag{159}$$

Additionally, we have to update equation (83) as follows:

$$\tilde{k}_H^* = \left(\frac{\tilde{c}_H^{e-\Psi}}{\tilde{c}_M^{e-1}}\frac{(1-a)\eta}{a(\lambda\beta^{-1} - 1 + \delta_H)}\right)^{\frac{1}{1-\psi}} = \left(\frac{\eta(1-\tau_h)\tilde{w}^*}{(1-\eta)(\lambda\beta^{-1} - 1 + \delta_H)h_H^{\Psi-1}\tilde{z}_H^{*\Psi}}\right)^{\frac{1}{1-\psi}} \tag{160}$$

Further, we have to replace equation (90) with the new definition of the general home production function.

$$\tilde{c}_H^* = [\eta\tilde{k}_H^{*\Psi} + (1-\eta)(\tilde{z}_H^* h_H)^{\Psi}]^{\frac{1}{\Psi}} \tag{161}$$

Finally, we have to adjust the expressions (92) and (94) for the parameters a and b.

$$a = [\eta^{-1}\tilde{c}_M^{*e-1}\tilde{c}_H^{*\Psi-e}\tilde{k}_H^{*1-\Psi}(\lambda\beta^{-1} - 1 + \delta_H) + 1]^{-1} \tag{162}$$

$$b = [(1-a)(1-\eta)\tilde{C}^{*-e}\tilde{c}_H^{*e-\Psi}h_H^{\Psi-1}z_H^{*\Psi}l + 1]^{-1} \tag{163}$$

The remaining calculations of the steady state remain unchanged compared with the standard model.

15.2.2 Summary

Compared with section 14, we replace equations (114), (116) and (123) with the updated expressions (153), (154) and (155).

$$\exp(\hat{c}_{Mt}) + \exp(\hat{x}_t) = \exp(\hat{y}_t) \tag{164}$$

$$\begin{aligned}(1-a)b(1-\eta)\exp[\hat{c}_{Ht}(e-\Psi) + \hat{z}_{Ht}\Psi + \hat{h}_{Ht}(\Psi-1) - \hat{C}_t e]\\ = (1-b)\exp(-\hat{l}_t)\end{aligned} \tag{165}$$

$$\begin{aligned}ab(1-\tau_h)(1-\theta)\exp(-\hat{C}_t e + \hat{c}_{Mt}(e-1) + \hat{y}_t - \hat{h}_{Mt})\\ = (1-b)\exp(-\hat{l}_t)\end{aligned} \tag{166}$$

$$\begin{aligned}\beta\exp(-\hat{C}_t e)\{a(1-\delta_H)\exp(\hat{c}_{Mt}(e-1)) + (1-a)\eta\exp[\hat{c}_{Ht}(e-\Psi) + \hat{k}_{Ht}(\Psi-1)]\}\\ = a\lambda\exp(-\hat{C}_{t-1}e + \hat{c}_{Mt-1}(e-1))\end{aligned} \tag{167}$$

$$\begin{aligned}\beta[\exp(\hat{r}_t)(1-\tau_k) + \delta_M\tau_k + 1 - \delta_M]\exp(-\hat{C}_t e + \hat{c}_{Mt}(e-1))\\ = \lambda\exp(-\hat{C}_{t-1}e + \hat{c}_{Mt-1}(e-1))\end{aligned} \tag{168}$$

$$\theta\exp(\hat{y}_t - \hat{k}_{Mt}) = \exp(\hat{r}_t) \tag{169}$$

$$(1-\theta)\exp(\hat{y}_t - \hat{h}_{Mt}) = \exp(\hat{w}_t) \tag{170}$$

$$\exp(\theta\hat{k}_{Mt} + (1-\theta)(\hat{z}_{Mt} + \hat{h}_{Mt})) = \exp(\hat{y}_t) \tag{171}$$

$$(a\exp(\hat{c}_{Mt}e) + (1-a)\exp(\hat{c}_{Ht}e))^{\frac{1}{e}} = \exp(\hat{C}_t) \tag{172}$$

$$1 - \exp(\hat{h}_{Ht}) - \exp(\hat{h}_{Mt}) = \exp(\hat{l}_t) \tag{173}$$

$$[\eta\exp(\hat{k}_{Ht}\Psi) + (1-\eta)\exp((\hat{z}_{Ht} + \hat{h}_{Ht})\Psi)]^{\frac{1}{\Psi}} = \exp(\hat{c}_{Ht}) \tag{174}$$

$$\lambda\exp(\hat{k}_{Mt+1}) - (1-\delta_M)\exp(\hat{k}_{Mt}) = \exp(\hat{x}_{Mt}) \tag{175}$$

$$\lambda\exp(\hat{k}_{Ht+1}) - (1-\delta_H)\exp(\hat{k}_{Ht}) = \exp(\hat{x}_{Ht}) \tag{176}$$

$$\exp(\hat{x}_{Mt}) + \exp(\hat{x}_{Ht}) = \exp(\hat{x}_t) \tag{177}$$

$$\exp(\hat{k}_{Mt}) + \exp(\hat{k}_{Ht}) = \exp(\hat{k}_t) \tag{178}$$

$$\exp(\hat{w}_t + \hat{h}_{Mt})\tau_h + \exp(\hat{r}_t + \hat{k}_{Mt})\tau_k - \delta_M\tau_k\exp(\hat{k}_{Mt}) = \exp(\hat{T}_t) \tag{179}$$

$$\rho_M\hat{z}_{Mt} + \epsilon_{Mt+1} = \hat{z}_{Mt+1} \tag{180}$$

$$\rho_H\hat{z}_{Ht} + \epsilon_{Ht+1} = \hat{z}_{Ht+1} \tag{181}$$

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